CHAPTER 2

Theoretical Basis for One-Dimensional Flow Calculations

This chapter describes the methodologies used in performing the one-dimensional flow calculations within HEC-RAS. The basic equations are presented along with discussions of the various terms. Solution schemes for the various equations are described. Discussions are provided as to how the equations should be applied, as well as applicable limitations.

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- General
- Steady Flow Water Surface Profiles
- Unsteady Flow Routing

General

This chapter describes the theoretical basis for one-dimensional water surface profile calculations. Discussions contained in this chapter are limited to steady flow water surface profile calculations and unsteady flow routing. When sediment transport calculations are added to the HEC-RAS system, discussions concerning this topic will be included in this manual.

Steady Flow Water Surface Profiles

HEC-RAS is currently capable of performing one-dimensional water surface profile calculations for steady gradually varied flow in natural or constructed channels. Subcritical, supercritical, and mixed flow regime water surface profiles can be calculated. Topics discussed in this section include: equations for basic profile calculations; cross section subdivision for conveyance calculations; composite Manning's n for the main channel; velocity weighting coefficient alpha; friction loss evaluation; contraction and expansion losses; computational procedure; critical depth determination; applications of the momentum equation; and limitations of the steady flow model.

Equations for Basic Profile Calculations

Water surface profiles are computed from one cross section to the next by solving the Energy equation with an iterative procedure called the standard step method. The Energy equation is written as follows:

$$Y_2 + Z_2 + \frac{\alpha_2 V_2^2}{2g} = Y_1 + Z_1 + \frac{\alpha_1 V_1^2}{2g} + h_e \qquad (2\text{-}1)$$

Where: Y_1, Y_2 = depth of water at cross sections

Z_1, Z_2 = elevation of the main channel inverts

V_1, V_2 = average velocities (total discharge/ total flow area)

α_1, α_2 = velocity weighting coefficients

g = gravitational acceleration

h_e = energy head loss

A diagram showing the terms of the energy equation is shown in Figure 2-1.

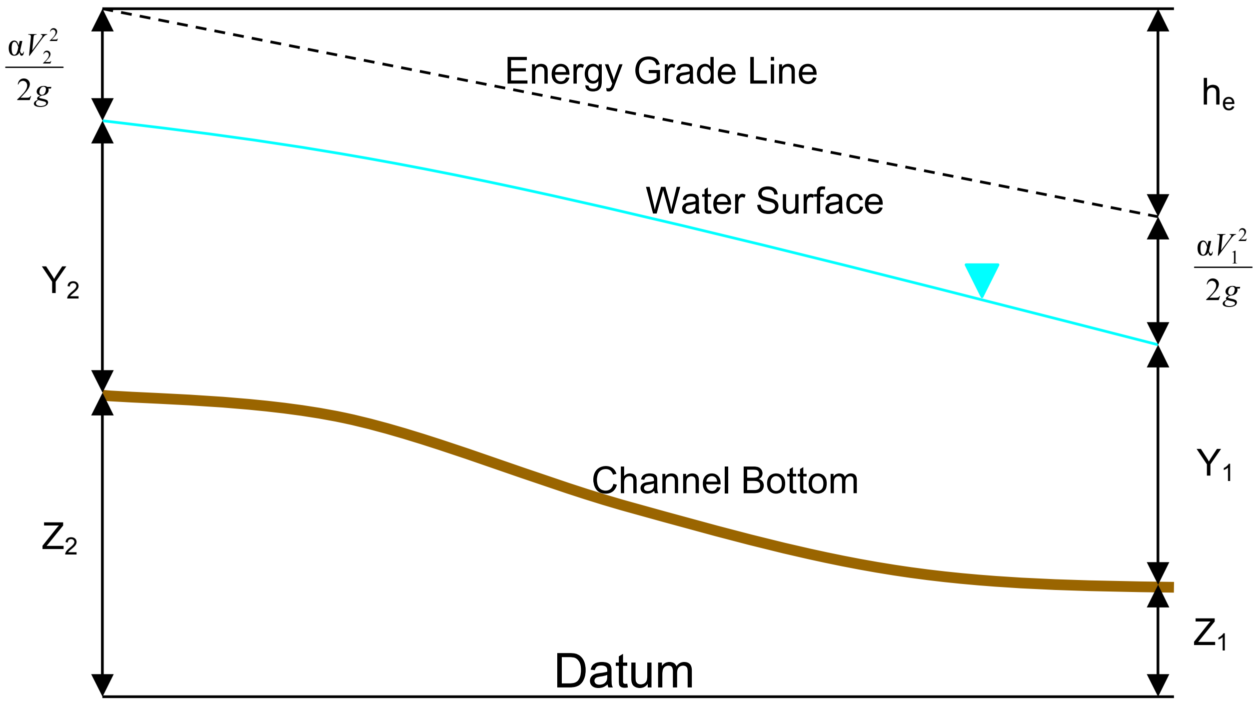


Figure 2.1 Representation of Terms in the Energy Equation

The energy head loss (h_e) between two cross sections is comprised of friction losses and contraction or expansion losses. The equation for the energy head loss is as follows:

$$h_e = L\overline{S}_f + C\left|\frac{\alpha_2 V_2^2}{2g} - \frac{\alpha_1 V_1^2}{2g}\right| \qquad (2\text{-}2)$$

Where: L = discharge weighted reach length

$\overline{S}_f$ = representative friction slope between two sections

C = expansion or contraction loss coefficient

The distance weighted reach length, L, is calculated as:

$$L = \frac{L_{lob}\,\overline{Q}_{lob} + L_{ch}\,\overline{Q}_{ch} + L_{rob}\,\overline{Q}_{rob}}{\overline{Q}_{lob} + \overline{Q}_{ch} + \overline{Q}_{rob}} \tag{2-3}$$

where: L_{lob}, L_{ch}, L_{rob} = cross section reach lengths specified for flow in the left overbank, main channel, and right overbank, respectively

$\overline{Q}_{lob}, \overline{Q}_{ch}, \overline{Q}_{rob}$ = arithmetic average of the flows between sections for the left overbank, main channel, and right overbank, respectively

Cross Section Subdivision for Conveyance Calculations

The determination of total conveyance and the velocity coefficient for a cross section requires that flow be subdivided into units for which the velocity is uniformly distributed. The approach used in HEC-RAS is to subdivide flow in the **overbank** areas using the input cross section n-value break points (locations where n-values change) as the basis for subdivision (Figure 2-2). Conveyance is calculated within each subdivision from the following form of Manning's equation (based on English units):

$$Q = K\,S_f^{1/2} \tag{2-4}$$

$$K = \frac{1.486}{n} A\,R^{2/3} \tag{2-5}$$

where: K = conveyance for subdivision

n = Manning's roughness coefficient for subdivision

A = flow area for subdivision

R = hydraulic radius for subdivision (area / wetted perimeter)

The program sums up all the incremental conveyances in the overbanks to obtain a conveyance for the left overbank and the right overbank. The main channel conveyance is normally computed as a single conveyance element. The total conveyance for the cross section is obtained by summing the three subdivision conveyances (left, channel, and right).

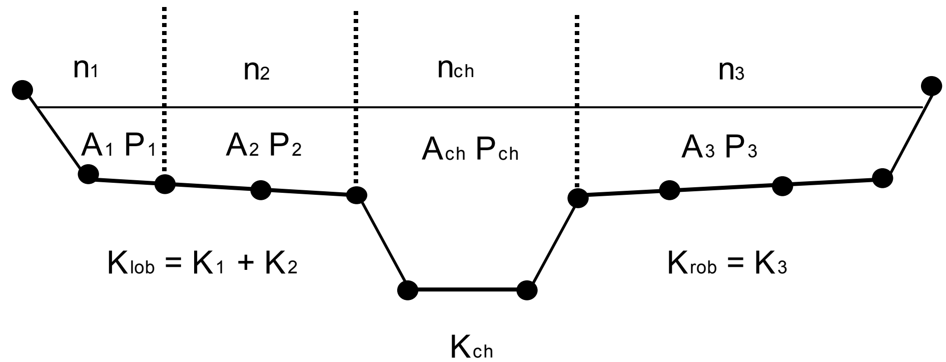


Figure 2.2 HEC-RAS Default Conveyance Subdivision Method

An alternative method available in HEC-RAS is to calculate conveyance between every coordinate point in the overbanks (Figure 2.3). The conveyance is then summed to get the total left overbank and right overbank values. This method is used in the Corps HEC-2 program. The method has been retained as an option within HEC-RAS in order to reproduce studies that were originally developed with HEC-2.

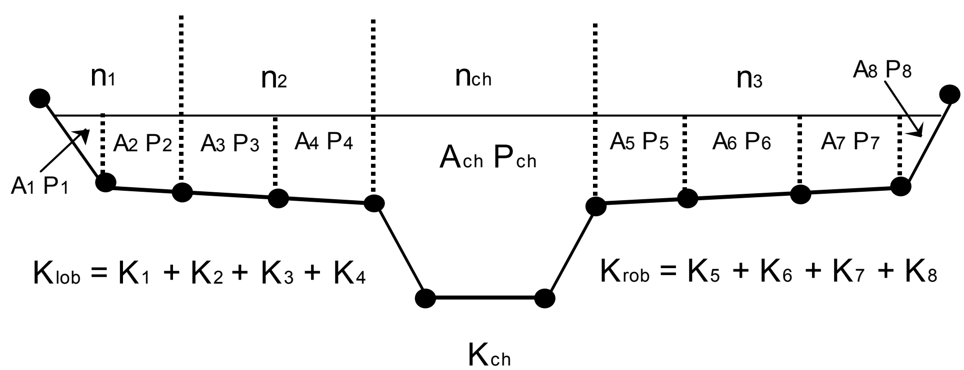


Figure 2.3 Alternative Conveyance Subdivision Method (HEC-2 style)

The two methods for computing conveyance will produce different answers whenever portions on the overbank have ground sections with significant vertical slopes. In general, the HEC-RAS default approach will provide a lower total conveyance for the same water surface elevation.

In order to test the significance of the two ways of computing conveyance, comparisons were performed using 97 data sets from the HEC profile accuracy study (HEC, 1986). Water surface profiles were computed for the 1% chance event using the two methods for computing conveyance in HEC-RAS. The results of the study showed that the HEC-RAS default approach will generally produce a higher computed water surface elevation. Out of the 2048 cross section locations, 47.5% had computed water surface elevations within 0.10 ft. (30.48 mm), 71% within 0.20 ft. (60.96 mm), 94.4% within 0.4 ft. (121.92 mm), 99.4% within 1.0 ft. (304.8 mm), and one cross section had a difference of 2.75 ft. (0.84 m). Because the differences tend to be in the same direction, some effects can be attributed to propagation of downstream differences.

The results from the conveyance comparisons do not show which method is more accurate, they only show differences. In general, it is felt that the HEC-RAS default method is more commensurate with the Manning equation and the concept of separate flow elements. Further research, with observed water surface profiles, will be needed to make any conclusions about the accuracy of the two methods.

Composite Manning's n for the Main Channel

Flow in the **main channel** is not subdivided, except when the roughness coefficient is changed within the channel area. HEC-RAS tests the applicability of subdivision of roughness within the main channel portion of a cross section, and if it is not applicable, the program will compute a single composite n value for the entire main channel. The program determines if the main channel portion of the cross section can be subdivided or if a composite main channel n value will be utilized based on the following criterion: if a main channel side slope is steeper than 5H:1V and the main channel has more than one n-value, a composite roughness n_c will be computed [Equation 6-17, Chow, 1959]. The channel side slope used by HEC-RAS is defined as the horizontal distance between adjacent n-value stations within the main channel over the difference in elevation of these two stations (see S_L and S_R of Figure 2.4).

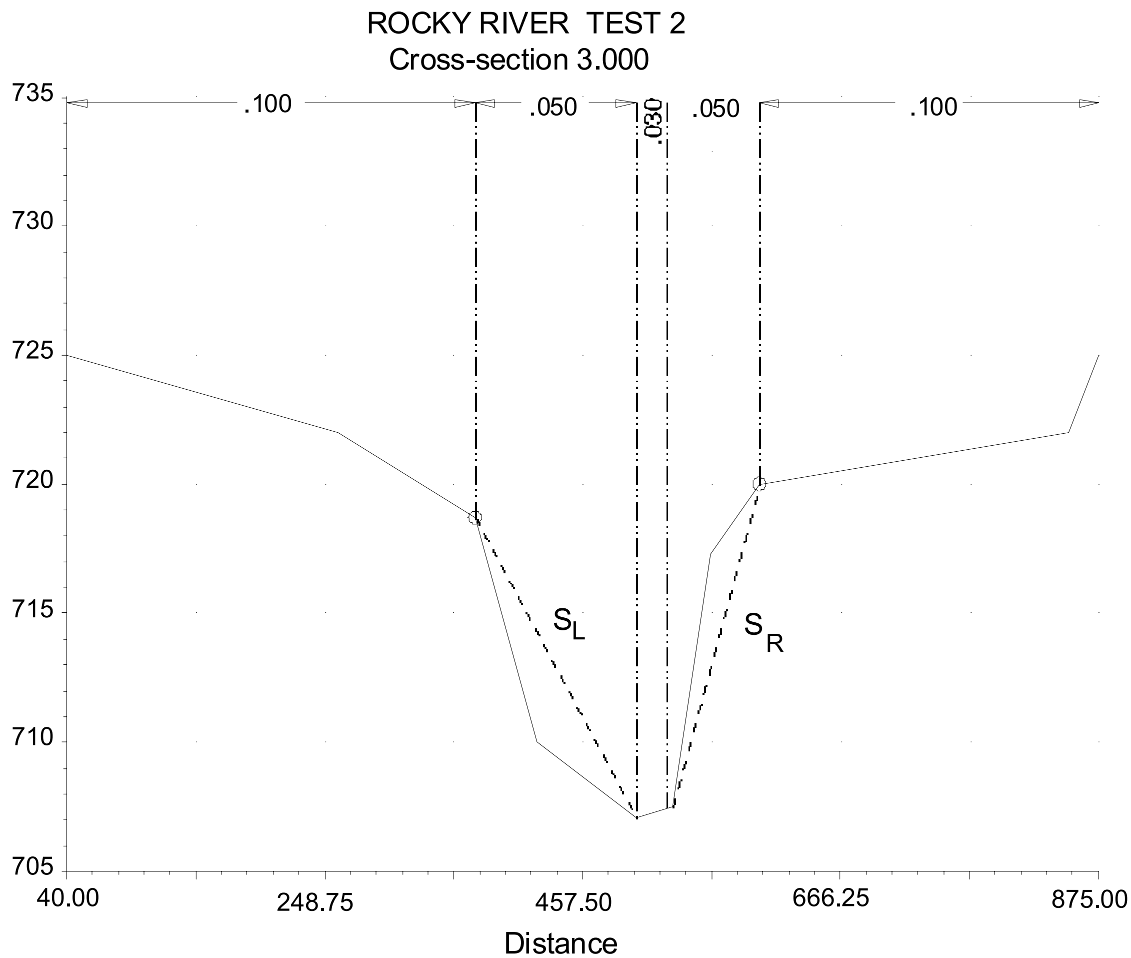


Figure 2.4 Definition of Bank Slope for Composite n_c Calculation

For the determination of n_c, the main channel is divided into N parts, each with a known wetted perimeter P_i and roughness coefficient n_i.

$$n_c = \left[\frac{\sum_{i=1}^{N} \left(P_i \, n_i^{1.5} \right)}{P} \right]^{2/3} \tag{2-6}$$

where: n_c = composite or equivalent coefficient of roughness

P = wetted perimeter of entire main channel

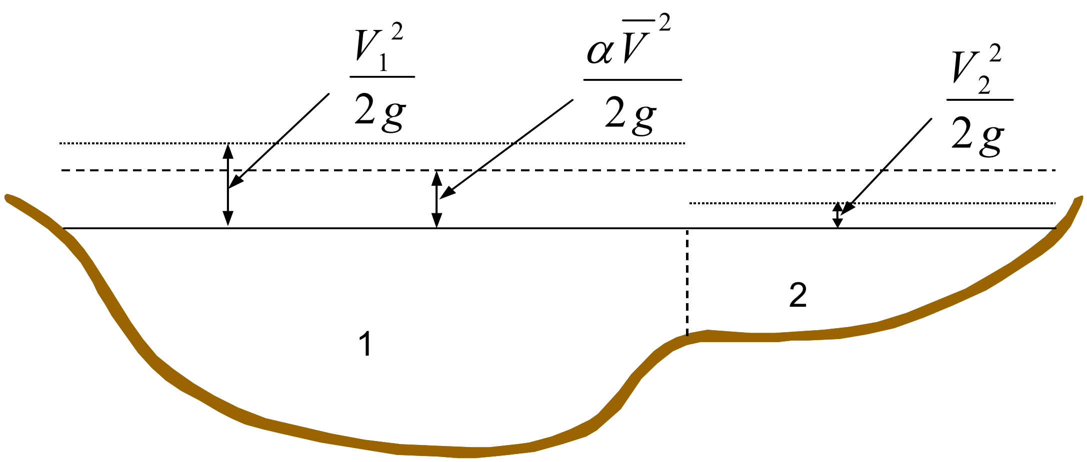
P_i = wetted perimeter of subdivision I

n_i = coefficient of roughness for subdivision I

The computed composite n_c should be checked for reasonableness. The computed value is the composite main channel *n* value in the output and summary tables.

Evaluation of the Mean Kinetic Energy Head

Because the HEC-RAS software is a one-dimensional water surface profiles program, only a single water surface and therefore a single mean energy are computed at each cross section. For a given water surface elevation, the mean energy is obtained by computing a flow weighted energy from the three subsections of a cross section (left overbank, main channel, and right overbank). Figure 2.5 below shows how the mean energy would be obtained for a cross section with a main channel and a right overbank (no left overbank area).



V_1 = mean velocity for subarea 1

V_2 = mean velocity for subarea 2

Figure 2.5 Example of How Mean Energy is Obtained

To compute the mean kinetic energy it is necessary to obtain the velocity head weighting coefficient alpha. Alpha is calculated as follows:

Mean Kinetic Energy Head = Discharge-Weighted Velocity Head

$$\alpha\frac{\overline{V}^2}{2g} = \frac{Q_1\frac{V_1^2}{2g} + Q_2\frac{V_2^2}{2g}}{Q_1 + Q_2} \tag{2-7}$$

$$\alpha = \frac{2g\left[Q_1 \frac{V_1^2}{2g} + Q_2 \frac{V_2^2}{2g}\right]}{(Q_1 + Q_2)\overline{V}^2} \tag{2-8}$$

$$\alpha = \frac{Q_1 V_1^2 + Q_2 V_2^2}{(Q_1 + Q_2)\overline{V}^2} \tag{2-9}$$

In General:

$$\alpha = \frac{\left[Q_1 V_1^2 + Q_2 V_2^2 + ... + Q_N V_N^2\right]}{Q\overline{V}^2} \tag{2-10}$$

The velocity coefficient, α, is computed based on the conveyance in the three flow elements: left overbank, right overbank, and channel. It can also be written in terms of conveyance and area as in the following equation:

$$\alpha = \frac{(A_t)^2\left[\frac{K_{lob}^3}{A_{lob}^2} + \frac{K_{ch}^3}{A_{ch}^2} + \frac{K_{rob}^3}{A_{rob}^2}\right]}{K_t^3} \tag{2-11}$$

Where: A_t = total flow area of cross section

A_{lob}, A_{ch}, A_{rob} = flow areas of left overbank, main channel and right overbank, respectively

K_t = total conveyance of cross section

K_{lob}, K_{ch}, K_{rob} = conveyances of left overbank, main channel and right overbank, respectively

Friction Loss Evaluation

Friction loss is evaluated in HEC-RAS as the product of $\overline{S}_f$ and L (Equation 2-2), where $\overline{S}_f$ is the representative friction slope for a reach and L is defined by Equation 2-3. The friction slope (slope of the energy gradeline) at each cross section is computed from Manning's equation as follows:

$$S_f = \left(\frac{Q}{K}\right)^2 \qquad (2\text{-}12)$$

Alternative expressions for the representative reach friction slope ($\overline{S}_f$) in HEC-RAS are as follows:

Average Conveyance Equation

$$\overline{S}_f = \left(\frac{Q_1 + Q_2}{K_1 + K_2}\right)^2 \qquad (2\text{-}13)$$

Average Friction Slope Equation

$$\overline{S}_f = \frac{S_{f1} + S_{f2}}{2} \qquad (2\text{-}14)$$

Geometric Mean Friction Slope Equation

$$\overline{S}_f = \sqrt{S_{f1} \times S_{f2}} \qquad (2\text{-}15)$$

Harmonic Mean Friction Slope Equation

$$\overline{S}_f = \frac{2\left(S_{f1} \times S_{f2}\right)}{S_{f1} + S_{f2}} \qquad (2\text{-}16)$$

Equation 2-13 is the "default" equation used by the program; that is, it is used automatically unless a different equation is requested by input. The program also contains an option to select equations, depending on flow regime and profile type (e.g., S1, M1, etc.). Further discussion of the alternative methods for evaluating friction loss is contained in Chapter 4, "Overview of Optional Capabilities."

Contraction and Expansion Loss Evaluation

Contraction and expansion losses in HEC-RAS are evaluated by the following equation:

$$h_{ce} = C\left|\frac{\alpha_1 V_1^2}{2g} - \frac{\alpha_2 V_2^2}{2g}\right| \tag{2-17}$$

Where: C = the contraction or expansion coefficient

The program assumes that a contraction is occurring whenever the velocity head downstream is greater than the velocity head upstream. Likewise, when the velocity head upstream is greater than the velocity head downstream, the program assumes that a flow expansion is occurring. Typical "C" values can be found in Chapter 3, "Basic Data Requirements."

Computation Procedure

The unknown water surface elevation at a cross section is determined by an iterative solution of Equations 2-1 and 2-2. The computational procedure is as follows:

1. Assume a water surface elevation at the upstream cross section (or downstream cross section if a supercritical profile is being calculated).

2. Based on the assumed water surface elevation, determine the corresponding total conveyance and velocity head.

3. With values from step 2, compute $\overline{S}_f$ and solve Equation 2-2 for h_e.

4. With values from steps 2 and 3, solve Equation 2-1 for WS_2.

5. Compare the computed value of WS_2 with the value assumed in step 1; repeat steps 1 through 5 until the values agree to within .01 feet (.003 m), or the user-defined tolerance.

The criterion used to assume water surface elevations in the iterative procedure varies from trial to trial. The first trial water surface is based on projecting the previous cross section's water depth onto the current cross section. The second trial water surface elevation is set to the assumed water surface elevation plus 70% of the error from the first trial (computed W.S. - assumed W.S.). In other words, W.S. new = W.S. assumed + 0.70 * (W.S. computed - W.S. assumed). The third and subsequent trials are generally based on a "Secant" method of projecting the rate of change of the difference between computed and assumed elevations for the previous two trials. The equation for the secant method is as follows:

$$WS_I = WS_{I-2} - Err_{I-2} * Err\_Assum / Err\_Diff \tag{2-18}$$

Where: WS_I = the new assumed water surface

WS_{I-1} = the previous iteration's assumed water surface

WS_{I-2} = the assumed water surface from two trials previous

Err_{I-2} = the error from two trials previous (computed water surface minus assumed from the I-2 iteration)

Err_Assum = the difference in assumed water surfaces from the previous two trials. Err_Assum = WS_{I-2} - WS_{I-1}

Err_Diff = the assumed water surface minus the calculated water surface from the previous iteration (I-1), plus the error from two trials previous (Err_{I-2}). Err_Diff = WS_{I-1} - WS_Calc_{I-1} + Err_{I-2}

The change from one trial to the next is constrained to a maximum of 50 percent of the assumed depth from the previous trial. On occasion the secant method can fail if the value of Err_Diff becomes too small. If the Err_Diff is less than 1.0E-2, then the secant method is not used. When this occurs, the program computes a new guess by taking the average of the assumed and computed water surfaces from the previous iteration.

The program is constrained by a *maximum number of iterations* (the default is 20) for balancing the water surface. While the program is iterating, it keeps track of the water surface that produces the minimum amount of error between the assumed and computed values. This water surface is called the *minimum error water surface*. If the maximum number of iterations is reached before a balanced water surface is achieved, the program will then calculate critical depth (if this has not already been done). The program then checks to see if the error associated with the *minimum error water surface* is within a predefined tolerance (the default is 0.3 ft or 0.1 m). If the minimum error water surface has an associated error less than the predefined tolerance, and this water surface is on the correct side of critical depth, then the program will use this water surface as the final answer and set a warning message that it has done so. If the minimum error water surface has an associated error that is greater than the predefined tolerance, or it is on the wrong side of critical depth, the program will use critical depth as the final answer for the cross section and set a warning message that it has done so. The rationale for using the minimum error water surface is that it is probably a better answer than critical depth, as long as the above criteria are met. Both the minimum error water surface and critical depth are only used in this situation to allow the program to continue the solution of the water surface profile. Neither of these two answers are considered to be valid solutions, and therefore warning messages are issued when either is used. In general, when the program cannot balance the energy equation at a cross section, it is usually caused by an inadequate number of cross sections (cross sections spaced too far apart) or bad cross section data. Occasionally, this can occur because the program is attempting to calculate a subcritical water surface when the flow regime is actually supercritical.

When a “balanced” water surface elevation has been obtained for a cross section, checks are made to ascertain that the elevation is on the “right” side of the critical water surface elevation (e.g., above the critical elevation if a subcritical profile has been requested by the user). If the balanced elevation is on the “wrong” side of the critical water surface elevation, critical depth is assumed for the cross section and a “warning” message to that effect is displayed by the program. The program user should be aware of critical depth assumptions and determine the reasons for their occurrence, because in many cases they result from reach lengths being too long or from misrepresentation of the effective flow areas of cross sections.

For a subcritical profile, a preliminary check for proper flow regime involves checking the Froude number. The program calculates the Froude number of the “balanced” water surface for both the main channel only and the entire cross section. If either of these two Froude numbers are greater than 0.94, then the program will check the flow regime by calculating a more accurate estimate of critical depth using the minimum specific energy method (this method is described in the next section). A Froude number of 0.94 is used instead of 1.0, because the calculation of Froude number in irregular channels is not accurate. Therefore, using a value of 0.94 is conservative, in that the program will calculate critical depth more often than it may need to.

For a supercritical profile, critical depth is automatically calculated for every cross section, which enables a direct comparison between balanced and critical elevations.

Critical Depth Determination

Critical depth for a cross section will be determined if any of the following conditions are satisfied:

(1) The supercritical flow regime has been specified.

(2) The calculation of critical depth has been requested by the user.

(3) This is an external boundary cross section and critical depth must be determined to ensure the user entered boundary condition is in the correct flow regime.

(4) The Froude number check for a subcritical profile indicates that critical depth needs to be determined to verify the flow regime associated with the balanced elevation.

(5) The program could not balance the energy equation within the

specified tolerance before reaching the maximum number of iterations.

The total energy head for a cross section is defined by:

$$H = WS + \frac{\alpha V^2}{2g} \qquad (2\text{-}19)$$

where: H = total energy head

WS = water surface elevation

$\frac{\alpha V^2}{2g}$ = velocity head

The critical water surface elevation is the elevation for which the total energy head is a minimum (i.e., minimum specific energy for that cross section for the given flow). The critical elevation is determined with an iterative procedure whereby values of WS are assumed and corresponding values of H are determined with Equation 2-19 until a minimum value for H is reached.

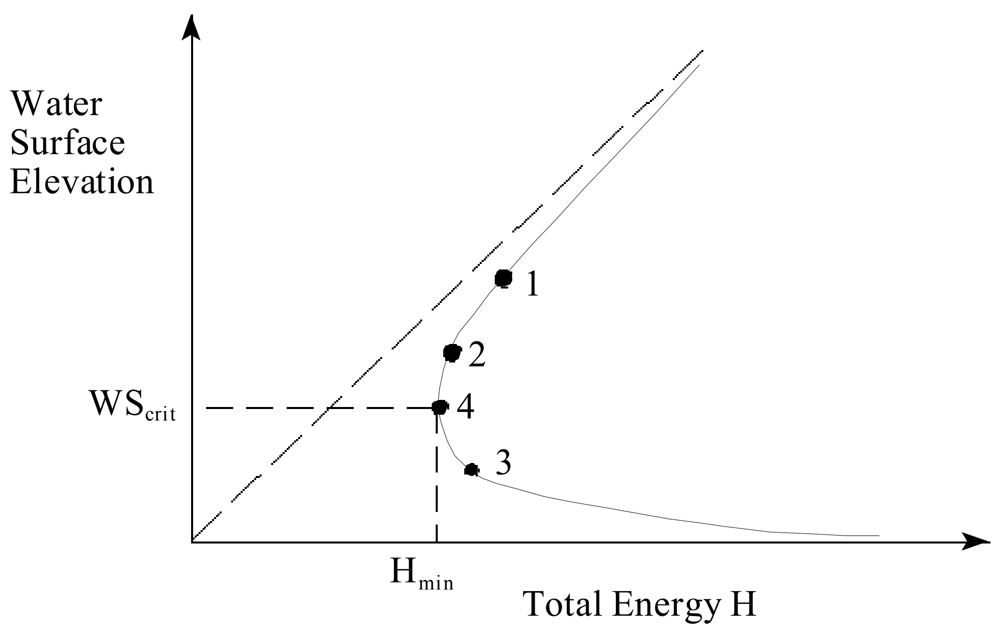


Figure 2.6 Energy vs. Water Surface Elevation Diagram

The HEC-RAS program has two methods for calculating critical depth: a “parabolic” method and a “secant” method. The parabolic method is computationally faster, but it is only able to locate a single minimum energy. For most cross sections there will only be one minimum on the total energy curve, therefore the parabolic method has been set as the default method (the default method can be changed from the user interface). If the parabolic method is tried and it does not converge, then the program will automatically try the secant method.

In certain situations it is possible to have more than one minimum on the total energy curve. Multiple minimums are often associated with cross sections that have breaks in the total energy curve. These breaks can occur due to very wide and flat overbanks, as well as cross sections with levees and ineffective flow areas. When the parabolic method is used on a cross section that has multiple minimums on the total energy curve, the method will converge on the first minimum that it locates. This approach can lead to incorrect estimates of critical depth. If the user thinks that the program has incorrectly located critical depth, then the secant method should be selected and the model should be re-simulated.

The "parabolic" method involves determining values of H for three values of WS that are spaced at equal ΔWS intervals. The WS corresponding to the minimum value for H, defined by a parabola passing through the three points on the H versus WS plane, is used as the basis for the next assumption of a value for WS. It is presumed that critical depth has been obtained when there is less than a 0.01 ft. (0.003 m) change in water depth from one iteration to the next and provided the energy head has not either decreased or increased by more than .01 feet (0.003 m).

The “secant” method first creates a table of water surface versus energy by slicing the cross section into 30 intervals. If the maximum height of the cross section (highest point to lowest point) is less than 1.5 times the maximum height of the main channel (from the highest main channel bank station to the invert), then the program slices the entire cross section into 30 equal intervals. If this is not the case, the program uses 25 equal intervals from the invert to the highest main channel bank station, and then 5 equal intervals from the main channel to the top of the cross section. The program then searches this table for the location of local minimums. When a point in the table is encountered such that the energy for the water surface immediately above and immediately below are greater than the energy for the given water surface, then the general location of a local minimum has been found. The program will then search for the local minimum by using the secant slope projection method. The program will iterate for the local minimum either thirty times or until the critical depth has been bounded by the critical error tolerance. After the local minimum has been determined more precisely, the program will continue searching the table to see if there are any other local minimums. The program can locate up to three local minimums in the energy curve. If more than one local minimum is found, the program sets critical depth equal to the one with the minimum energy. If this local minimum is due to a break in the energy curve caused by overtopping a levee or an ineffective flow area, then the program will select the next lowest minimum on the energy curve. If all of the local minimums are occurring at breaks in the energy curve (caused by levees and ineffective flow areas), then the program will set critical depth to the one with the lowest energy. If no local minimums are found, then the program will use the water surface elevation with the least energy. If the critical depth that is found is at the top of the cross section, then this is probably not a real critical depth. Therefore, the program will double the height of the cross section and try again. Doubling the height of the cross section is accomplished by extending vertical walls at the first and last points of the section. The height of the cross section can be doubled five times before the program will quit searching.

Applications of the Momentum Equation

Whenever the water surface passes through critical depth, the energy equation is not considered to be applicable. The energy equation is only applicable to gradually varied flow situations, and the transition from subcritical to supercritical or supercritical to subcritical is a rapidly varying flow situation. There are several instances when the transition from subcritical to supercritical and supercritical to subcritical flow can occur. These include significant changes in channel slope, bridge constrictions, drop structures and weirs, and stream junctions. In some of these instances empirical equations can be used (such as at drop structures and weirs), while at others it is necessary to apply the momentum equation in order to obtain an answer.

Within HEC-RAS, the momentum equation can be applied for the following specific problems: the occurrence of a hydraulic jump; low flow hydraulics at bridges; and stream junctions. In order to understand how the momentum equation is being used to solve each of the three problems, a derivation of the momentum equation is shown here. The application of the momentum equation to hydraulic jumps and stream junctions is discussed in detail in Chapter 4. Detailed discussions on applying the momentum equation to bridges is discussed in Chapter 5.

The momentum equation is derived from Newton's second law of motion:

Force = Mass x Acceleration (change in momentum)

$$\sum F_x = m\,a \qquad (2\text{-}20)$$

Applying Newton's second law of motion to a body of water enclosed by two cross sections at locations 1 and 2 (Figure 2.7), the following expression for the change in momentum over a unit time can be written:

$$P_2 - P_1 + W_x - F_f = Q\,\rho\,\Delta V_x \qquad (2\text{-}21)$$

Where: P = Hydrostatic pressure force at locations 1 and 2.
W_x = Force due to the weight of water in the X direction.
F_f = Force due to external friction losses from 2 to 1.
Q = Discharge.
ρ = Density of water
ΔV_x = Change in velocity from 2 to 1, in the X direction.

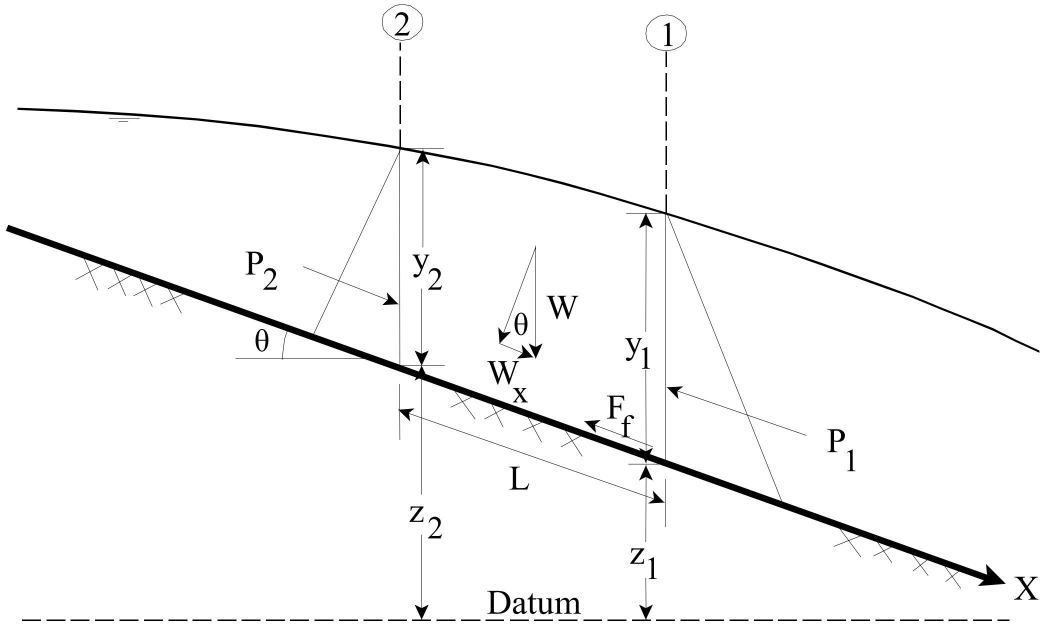


Figure 2.7 Application of the Momentum Principle

Hydrostatic Pressure Forces:

The force in the X direction due to hydrostatic pressure is:

$$P = \gamma A \overline{Y} \cos\theta \tag{2-22}$$

The assumption of a hydrostatic pressure distribution is only valid for slopes less than 1:10. The cos θ for a slope of 1:10 (approximately 6 degrees) is equal to 0.995. Because the slope of ordinary channels is far less than 1:10, the cos θ correction for depth can be set equal to 1.0 (Chow, 1959). Therefore, the equations for the hydrostatic pressure force at sections 1 and 2 are as follows:

$$P_1 = \gamma A_1 \overline{Y}_1 \tag{2-23}$$

$$P_2 = \gamma A_2 \overline{Y}_2 \tag{2-24}$$

Where: γ = Unit weight of water
A_i = Wetted area of the cross section at locations 1 and 2
$\overline{Y}_i$ = Depth measured from the water surface to the centroid of the cross sectional area at locations 1 and 2

Weight of Water Force:

Weight of water = (unit weight of water) x (volume of water)

$$W = \gamma \left(\frac{A_1 + A_2}{2} \right) L \tag{2-25}$$

$$W_x = W \times \sin\theta \tag{2-26}$$

$$\sin\theta = \frac{z_2 - z_1}{L} = S_0 \tag{2-27}$$

$$W_x = \gamma \left(\frac{A_1 + A_2}{2} \right) L\, S_0 \tag{2-28}$$

Where: L = Distance between sections 1 and 2 along the X axis
S_o = Slope of the channel, based on mean bed elevations
z_{i} = Mean bed elevation at locations 1 and 2

Force of External Friction:

$$F_f = \tau \overline{P} L \tag{2-29}$$

Where: τ = Shear stress

$\overline{P}$ = Average wetted perimeter between sections 1 and 2

$$\tau = \gamma \overline{R} \overline{S}_f \tag{2-30}$$

Where: $\overline{R}$ = Average hydraulic radius (R = A/P)

$\overline{S}_f$ = Slope of the energy grade line (friction slope)

$$F_f = \gamma \frac{\overline{A}}{\overline{P}} \overline{S}_f \overline{P} L \tag{2-31}$$

$$F_f = \gamma \left(\frac{A_1 + A_2}{2} \right) \overline{S}_f \; L \tag{2-32}$$

Mass times Acceleration:

$$m a = Q \rho \Delta V_x \tag{2-33}$$

$$\rho = \frac{\gamma}{g} \quad and \quad \Delta V_x = \left(\beta_1 V_1 - \beta_2 V_2 \right)$$

$$m a = \frac{Q \gamma}{g} \left(\beta_1 V_1 - \beta_2 V_2 \right) \tag{2-34}$$

Where: β = momentum coefficient that accounts for a varying velocity distribution in irregular channels

Substituting Back into Equation 2-21, and assuming Q can vary from 2 to 1:

$$\gamma A_2 \overline{Y}_2 - \gamma A_1 \overline{Y}_1 + \gamma\left(\frac{A_1 + A_2}{2}\right) L S_0 - \gamma\left(\frac{A_1 + A_2}{2}\right) L \overline{S}_f = \frac{Q_1 \gamma}{g} \beta_1 V_1 - \frac{Q_2 \gamma}{g} \beta_2 V_2 \tag{2-35}$$

$$\frac{Q_2 \beta_2 V_2}{g} + A_2 \overline{Y}_2 + \left(\frac{A_1 + A_2}{2}\right) L S_0 - \left(\frac{A_1 + A_2}{2}\right) L \overline{S}_f = \frac{Q_1 \beta_1 V_1}{g} + A_1 \overline{Y}_1 \tag{2-36}$$

$$\frac{Q_2 \beta_2}{g A_2} + A_2 \overline{Y}_2 + \left(\frac{A_1 + A_2}{2}\right) L S_0 - \left(\frac{A_1 + A_2}{2}\right) L \overline{S}_f = \frac{Q_1 \beta_1}{g A_1} + A_1 \overline{Y}_1 \tag{2-37}$$

Equation 2-37 is the functional form of the momentum equation that is used in HEC-RAS. All applications of the momentum equation within HEC-RAS are derived from equation 2-37.

Air Entrainment in High Velocity Streams

For channels that have high flow velocity, the water surface may be slightly higher than otherwise expected due to the entrainment of air. While air entrainment is not important for most rivers, it can be significant for highly supercritical flows (Froude numbers greater than 1.6). HEC-RAS now takes this into account with the following two equations (EM 1110-2-1601, plate B-50):

For Froude numbers less than or equal to 8.2,

$$D_a = 0.906\, D\left(e\right)^{0.061F} \tag{2-38}$$

For Froude numbers greater than 8.2,

$$D_a = 0.620\, D\left(e\right)^{0.1051F} \tag{2-39}$$

Where: Da = water depth with air entrainment
D = water depth without air entrainment
e = numerical constant, equal to 2.718282
F = Froude number

A water surface with air entrainment is computed and displayed separately in the HEC-RAS tabular output. In order to display the water surface with air entrainment, the user must create their own profile table and include the variable "WS Air Entr." within that table. This variable is not automatically displayed in any of the standard HEC-RAS tables.

Steady Flow Program Limitations

The following assumptions are implicit in the analytical expressions used in the current version of the program:

(1) Flow is steady.

(2) Flow is gradually varied. (Except at hydraulic structures such as: bridges; culverts; and weirs. At these locations, where the flow can be rapidly varied, the momentum equation or other empirical equations are used.)

(3) Flow is one dimensional (i.e., velocity components in directions other than the direction of flow are not accounted for).

(4) River channels have "small" slopes, say less than 1:10.

Flow is assumed to be steady because time-dependent terms are not included in the energy equation (Equation 2-1). Flow is assumed to be gradually varied because Equation 2-1 is based on the premise that a hydrostatic pressure distribution exists at each cross section. At locations where the flow is rapidly varied, the program switches to the momentum equation or other empirical equations. Flow is assumed to be one-dimensional because Equation 2-19 is based on the premise that the total energy head is the same for all points in a cross section. Small channel slopes are assumed because the pressure head, which is a component of Y in Equation 2-1, is represented by the water depth measured vertically.

The program does not currently have the capability to deal with movable boundaries (i.e., sediment transport) and requires that energy losses be definable with the terms contained in Equation 2-2.

Unsteady Flow Routing

The physical laws which govern the flow of water in a stream are: (1) the principle of conservation of mass (continuity), and (2) the principle of conservation of momentum. These laws are expressed mathematically in the form of partial differential equations, which will hereafter be referred to as the continuity and momentum equations. The derivations of these equations are presented in this chapter based on a paper by James A. Liggett from the book "Unsteady Flow in Open Channels" (Mahmmod and Yevjevich, 1975).

Continuity Equation

Consider the elementary control volume shown in Figure 2.8. In this figure, distance *x* is measured along the channel, as shown. At the midpoint of the control volume the flow and total flow area are denoted *Q(x,t)* and A_T, respectively. The total flow area is the sum of active area *A* and off-channel storage area *S*.

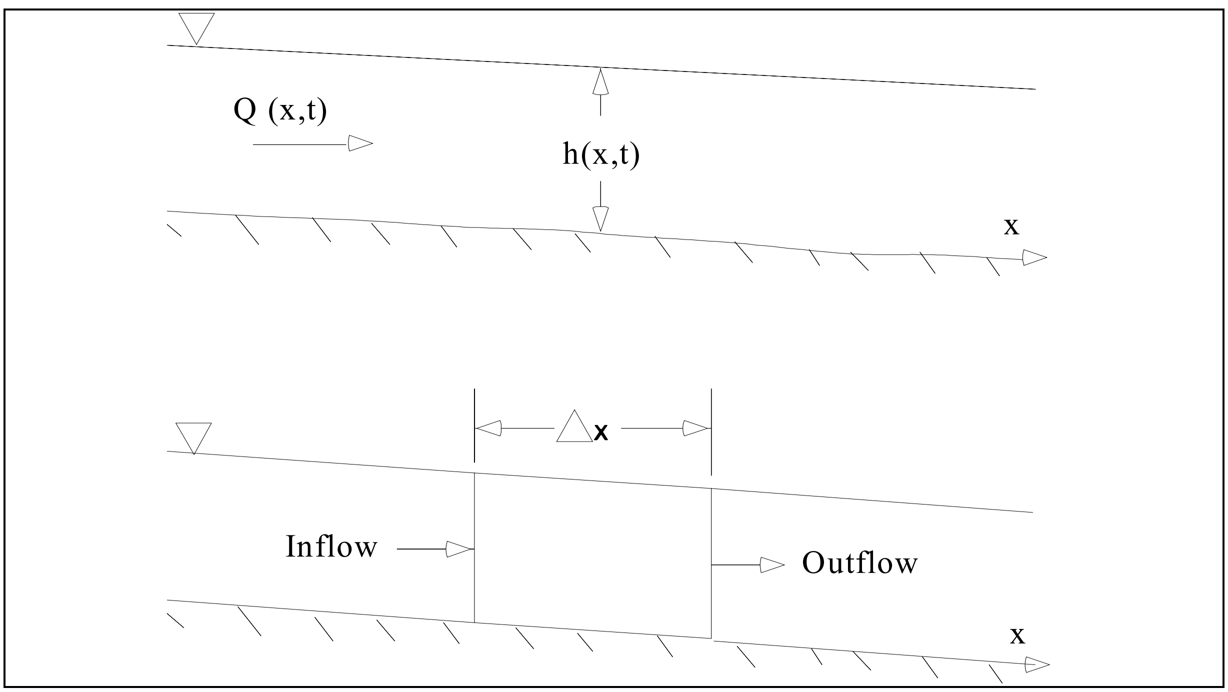


Figure 2.8 Elementary Control Volume for Derivation of Continuity and Momentum Equations.

Conservation of mass for a control volume states that *the net rate of flow into the volume be equal to the rate of change of storage inside the volume*. The rate of inflow to the control volume may be written as:

$$Q - \frac{\partial Q}{\partial x}\frac{\Delta x}{2} \tag{2-40}$$

the rate of outflow as:

$$Q + \frac{\partial Q}{\partial x}\frac{\Delta x}{2} \tag{2-41}$$

and the rate of change in storage as:

$$\frac{\partial A_T}{\partial t}\Delta x \tag{2-42}$$

Assuming that Δx is small, the change in mass in the control volume is equal to:

$$\rho\frac{\partial A_T}{\partial t}\Delta x = \rho\left[\left(Q - \frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right) - \left(Q + \frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right) + Q_l\right] \tag{2-43}$$

where Q_l is the lateral flow entering the control volume and ρ is the fluid density. Simplifying and dividing through by $\rho\Delta x$ yields the final form of the continuity equation:

$$\frac{\partial A_T}{\partial t} + \frac{\partial Q}{\partial x} - q_l = 0 \tag{2-44}$$

in which q_l is the lateral inflow per unit length.

Momentum Equation

Conservation of momentum is expressed by Newton's second law as:

$$\sum F_x = \frac{d\vec{M}}{dt} \tag{2-45}$$

Conservation of momentum for a control volume states that *the net rate of momentum entering the volume* (momentum flux) *plus the sum of all external forces acting on the volume be equal to the rate of accumulation of momentum*. This is a vector equation applied in the *x*-direction. The momentum flux (MV) is the fluid mass times the velocity vector in the direction of flow. Three forces will be considered: (1) pressure, (2) gravity and (3) boundary drag, or friction force.

Pressure forces: Figure 2.9 illustrates the general case of an irregular cross section. The pressure distribution is assumed to be hydrostatic (pressure varies linearly with depth) and the total pressure force is the integral of the pressure-area product over the cross section. After Shames (1962), the pressure force at any point may be written as:

$$F_P = \int_0^h \rho\, g\,(h - y)\, T(y)\, dy \tag{2-46}$$

where h is the depth, y the distance above the channel invert, and $T(y)$ a width function which relates the cross section width to the distance above the channel invert.

If F_p is the pressure force in the *x*-direction at the midpoint of the control volume, the force at the upstream end of the control volume may be written as:

$$F_P - \frac{\partial F_P}{\partial x}\frac{\Delta x}{2} \tag{2-47}$$

and at the downstream end as:

$$F_P + \frac{\partial F_P}{\partial x}\frac{\Delta x}{2} \tag{2-48}$$

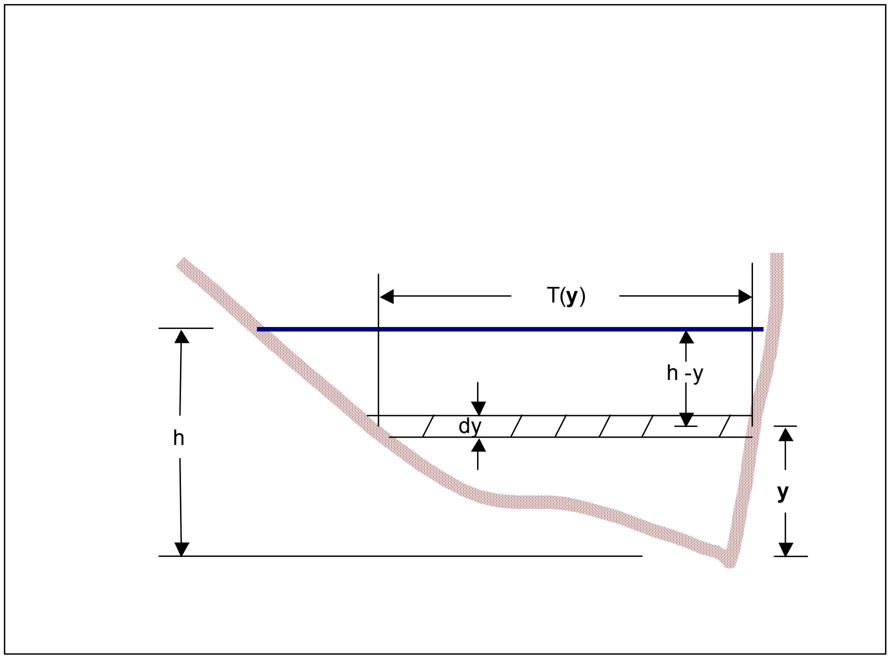


Figure 2.9 Illustration of Terms Associated with Definition of Pressure Force.

The sum of the pressure forces for the control volume may therefore be written as:

$$F_{Pn} = \left| F_P - \frac{\partial F_P}{\partial x}\frac{\Delta x}{2} \right| - \left| F_P + \frac{\partial F_P}{\partial x}\frac{\Delta x}{2} \right| + F_B \qquad \text{(2-49)}$$

where F_{Pn} is the net pressure force for the control volume, and F_B is the force exerted by the banks in the *x*-direction on the fluid. This may be simplified to:

$$F_{Pn} = -\frac{\partial F_P}{\partial x}\Delta x + F_B \qquad \text{(2-50)}$$

Differentiating equation 2-46 using Leibnitz's Rule and then substituting in equation 2-50 results in:

$$F_{Pn} = -\rho\, g \Delta x \left[\frac{\partial h}{\partial x} \int_0^h T(y) dy + \int_0^h (h-y) \frac{\partial T(y)}{\partial x} dy \right] + F_B \qquad (2\text{-}51)$$

The first integral in equation 2-51 is the cross-sectional area, A. The second integral (multiplied by $-\rho g \Delta x$) is the pressure force exerted by the fluid on the banks, which is exactly equal in magnitude, but opposite in direction to F_B. Hence the net pressure force may be written as:

$$F_{Pn} = -\rho\, g\, A \frac{\partial h}{\partial x} \Delta x \qquad (2\text{-}52)$$

Gravitational force: The force due to gravity on the fluid in the control volume in the *x*-direction is:

$$F_g = \rho\, gA \sin\theta\ \Delta x \qquad (2\text{-}53)$$

here θ is the angle that the channel invert makes with the horizontal. For natural rivers θ is small and $\sin\theta \approx \tan\theta = -\partial Z_0 / \partial X$, where z_0 is the invert elevation. Therefore the gravitational force may be written as:

$$F_g = -\rho g A \frac{\partial z_0}{\partial x} \Delta x \qquad (2\text{-}54)$$

This force will be positive for negative bed slopes.

Boundary drag (friction force): Frictional forces between the channel and the fluid may be written as:

$$F_f = -\tau_0 P \Delta x \qquad (2\text{-}55)$$

where τ_o is the average boundary shear stress (force/unit area) acting on the fluid boundaries, and *P* is the wetted perimeter. The negative sign indicates that, with flow in the positive *x*-direction, the force acts in the negative x-direction. From dimensional analysis, τ_o may be expressed in terms of a drag coefficient, C_D, as follows:

$$\boldsymbol{\tau}_0 = \boldsymbol{\rho} C_D V^2 \qquad (2\text{-}56)$$

The drag coefficient may be related to the Chezy coefficient, *C*, by the following:

$$C_D = \frac{g}{C^2} \tag{2-57}$$

Further, the Chezy equation may be written as:

$$V = C\sqrt{RS_f} \tag{2-58}$$

Substituting equations 2-56, 2-57, and 2-58 into 2-55, and simplifying, yields the following expression for the boundary drag force:

$$F_f = -\rho\, g\, A\, S_f\, \Delta x \tag{2-59}$$

where S_f is the friction slope, which is positive for flow in the positive *x*-direction. The friction slope must be related to flow and stage. Traditionally, the Manning and Chezy friction equations have been used. Since the Manning equation is predominantly used in the United States, it is also used in HEC-RAS. The Manning equation is written as:

$$S_f = \frac{Q|Q|n^2}{2.208 R^{4/3} A^2} \tag{2-60}$$

where *R* is the hydraulic radius and *n* is the Manning friction coefficient.

Momentum flux: With the three force terms defined, only the momentum flux remains. The flux entering the control volume may be written as:

$$\rho\left[QV - \frac{\partial QV}{\partial x}\frac{\Delta x}{2}\right] \tag{2-61}$$

and the flux leaving the volume may be written as:

$$\rho\left[QV + \frac{\partial QV}{\partial x}\frac{\Delta x}{2}\right] \tag{2-62}$$

Therefore the net rate of momentum (momentum flux) entering the control volume is:

$$-\rho\frac{\partial QV}{\partial x}\Delta x \tag{2-63}$$

Since the momentum of the fluid in the control volume is ρQΔx, the rate of accumulation of momentum may be written as:

$$\frac{\partial}{\partial t}\left(\rho Q\Delta x\right)=\rho\Delta x\frac{\partial Q}{\partial t} \tag{2-64}$$

Restating the principle of conservation of momentum:

The net rate of momentum (momentum flux) *entering the volume* (2-63) *plus the sum of all external forces acting on the volume* [(2-52) + (2-54) + (2-59)] *is equal to the rate of accumulation of momentum* (2-64). Hence:

$$\rho\Delta x\frac{\partial Q}{\partial t}=-\rho\frac{\partial QV}{\partial x}\Delta x-\rho gA\frac{\partial h}{\partial x}\Delta x-\rho gA\frac{\partial z_0}{\partial x}\Delta x-\rho gAS_f\Delta x \tag{2-65}$$

The elevation of the water surface, *z*, is equal to $z_0 + h$. Therefore:

$$\frac{\partial z}{\partial x}=\frac{\partial h}{\partial x}+\frac{\partial z_0}{\partial x} \tag{2-66}$$

where $\partial z / \partial x$ is the water surface slope. Substituting (2-66) into (2-65), dividing through by $\rho\Delta x$ and moving all terms to the left yields the final form of the momentum equation:

$$\frac{\partial Q}{\partial t}+\frac{\partial QV}{\partial x}+gA\left(\frac{\partial z}{\partial x}+S_f\right)=0 \tag{2-67}$$

Application of the Unsteady Flow Equations Within HEC-RAS

Figure 2-10 illustrates the two-dimensional characteristics of the interaction between the channel and floodplain flows. When the river is rising water moves laterally away from the channel, inundating the floodplain and filling available storage areas. As the depth increases, the floodplain begins to convey water downstream generally along a shorter path than that of the main channel. When the river stage is falling, water moves toward the channel from the overbank supplementing the flow in the main channel.

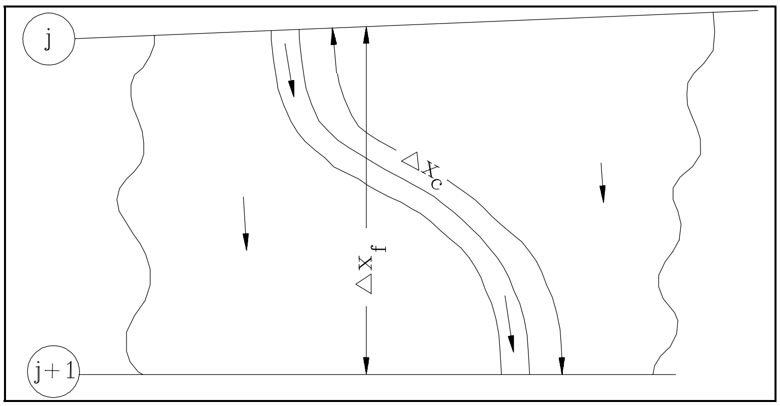


Figure 2.10 Channel and floodplain flows

Because the primary direction of flow is oriented along the channel, this two-dimensional flow field can often be accurately approximated by a one-dimensional representation. Off-channel ponding areas can be modeled with storage areas that exchange water with the channel. Flow in the overbank can be approximated as flow through a separate channel.

This channel/floodplain problem has been addressed in many different ways. A common approach is to ignore overbank conveyance entirely, assuming that the overbank is used only for storage. This assumption may be suitable for large streams such as the Mississippi River where the channel is confined by levees and the remaining floodplain is either heavily vegetated or an off-channel storage area. Fread (1976) and Smith (1978) approached this problem by dividing the system into two separate channels and writing continuity and momentum equations for each channel. To simplify the problem they assumed a horizontal water surface at each cross section normal to the direction of flow; such that the exchange of momentum between the channel and the floodplain was negligible and that the discharge was distributed according to conveyance, i.e.:

$$Q_c = \phi \; Q \qquad (2\text{-}68)$$

Where: Q_c = flow in channel,
Q = total flow,
ϕ = $K_c / (K_c + K_f)$,
K_c = conveyance in the channel, and,
K_f = conveyance in the floodplain.

With these assumptions, the one-dimensional equations of motion can be combined into a single set:

$$\frac{\partial A}{\partial t} + \frac{\partial (\Phi Q)}{\partial x_c} + \frac{\partial [(1-\Phi)Q]}{\partial x_f} = 0 \qquad (2\text{-}69)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial(\Phi^2 Q^2 / A_c)}{\partial x_c} + \frac{\partial((1-\Phi)^2 Q^2 / A_f)}{\partial x_f} + gA_c\left[\frac{\partial Z}{\partial x_c} + S_{fc}\right] + gA_f\left[\frac{\partial z}{\partial x_f} + S_{ff}\right] = 0 \qquad (2\text{-}70)$$

in which the subscripts *c* and *f* refer to the channel and floodplain, respectively. These equations were approximated using implicit finite differences, and solved numerically using the Newton-Raphson iteration technique. The model was successful and produced the desired effects in test problems. Numerical oscillations, however, can occur when the flow at one node, bounding a finite difference cell, is within banks and the flow at the other node is not.

Expanding on the earlier work of Fread and Smith, Barkau (1982) manipulated the finite difference equations for the channel and floodplain and defined a new set of equations that were computationally more convenient. Using a velocity distribution factor, he combined the convective terms. Further, by defining an equivalent flow path, Barkau replaced the friction slope terms with an equivalent force.

The equations derived by Barkau are the basis for the unsteady flow solution within the HEC-RAS software. These equations were derived above. The numerical solution of these equations is described in the next sections.

Implicit Finite Difference Scheme

The most successful and accepted procedure for solving the one-dimensional unsteady flow equations is the four-point implicit scheme, also known as the box scheme (Figure 2.11). Under this scheme, space derivatives and function values are evaluated at an interior point, $(n+\theta)\ \Delta t$. Thus values at $(n+1)\ \Delta t$ enter into all terms in the equations. For a reach of river, a system of simultaneous equations results. The simultaneous solution is an important aspect of this scheme because it allows information from the entire reach to influence the solution at any one point. Consequently, the time step can be significantly larger than with explicit numerical schemes. Von Neumann stability analyses performed by Fread (1974), and Liggett and Cunge (1975), show the implicit scheme to be unconditionally stable (theoretically) for $0.5 < \theta \leq 1.0$, conditionally stable for $\theta = 0.5$, and unstable for $\theta < 0.5$. In a convergence analysis performed by the same authors, it was shown that numerical damping increased as the ratio $\lambda/\Delta x$ decreased, where λ is the length of a wave in the hydraulic system. For streamflow routing problems where the wavelengths are long with respect to spatial distances, convergence is not a serious problem.

In practice, other factors may also contribute to the non-stability of the solution scheme. These factors include dramatic changes in channel cross-sectional properties, abrupt changes in channel slope, characteristics of the flood wave itself, and complex hydraulic structures such as levees, bridges, culverts, weirs, and spillways. In fact, these other factors often overwhelm any stability considerations associated with θ. **Because of these factors, any model application should be accompanied by a sensitivity study, where the accuracy and the stability of the solution are tested with various time and distance intervals.**

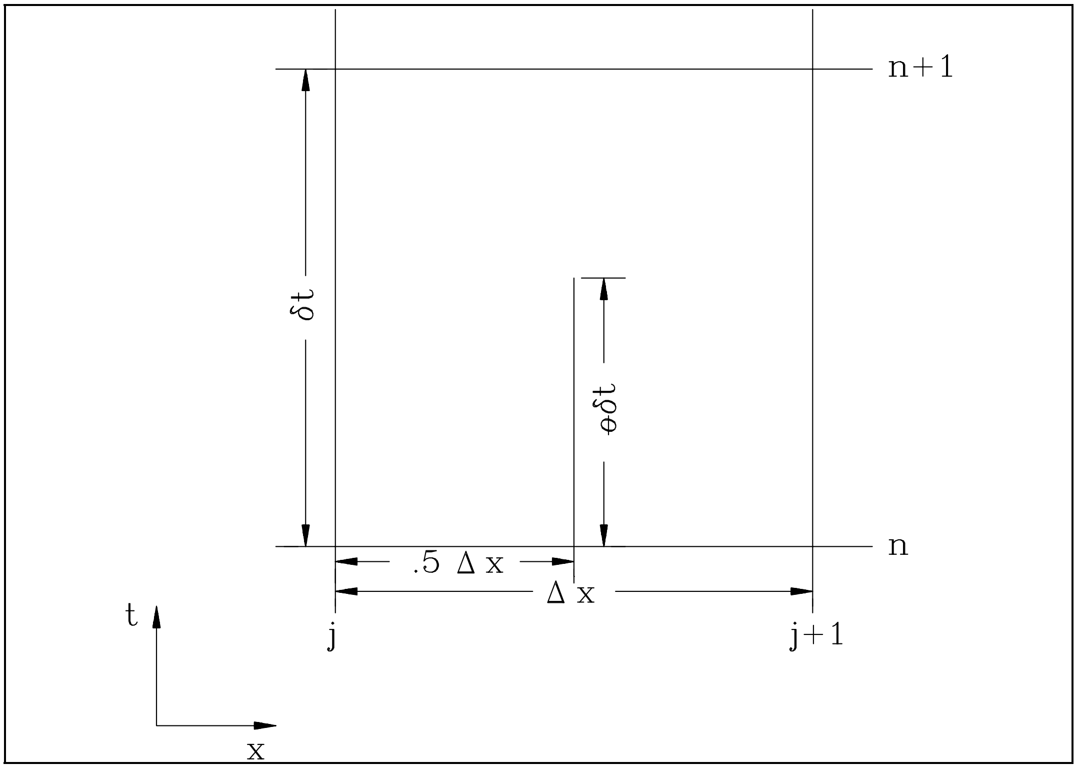


Figure 2.11 Typical finite difference cell.

The following notation is defined:

$$f_j = f_j^n \tag{2-71}$$

and:

$$\Delta f_j = f_j^{n+1} - f_j^n \tag{2-72}$$

then:

$$f_j^{n+1} = f_j + \Delta f_j \tag{2-73}$$

The general implicit finite difference forms are:

1. Time derivative

$$\frac{\partial f}{\partial t} \approx \frac{\Delta f}{\Delta t} = \frac{0.5\,(\Delta f_{j+1} + \Delta f_j)}{\Delta t} \tag{2-74}$$

2. Space derivative

$$\frac{\partial f}{\partial x} \approx \frac{\Delta f}{\Delta x} = \frac{(f_{j+1} - f_j) + \theta\,(\Delta f_{j+1} - \Delta f_j)}{\Delta x} \tag{2-75}$$

3. Function value

$$f \approx \bar{f} = 0.5\,(f_j + f_{j+1}) + 0.5\,\theta\,(\Delta f_j + \Delta f_{j+1}) \qquad \text{(2-76)}$$

Continuity Equation

The continuity equation describes conservation of mass for the one-dimensional system. From previous text, with the addition of a storage term, S, the continuity equation can be written as:

$$\frac{\partial A}{\partial t} + \frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} - q_l = 0 \qquad \text{(2-77)}$$

where:

	x	=	distance along the channel,
	t	=	time,
	Q	=	flow,
	A	=	cross-sectional area,
	S	=	storage from non conveying portions of cross section,
	q_l	=	lateral inflow per unit distance.

The above equation can be written for the channel and the floodplain:

$$\frac{\partial Q_c}{\partial x_c} + \frac{\partial A_c}{\partial t} = q_f \qquad \text{(2-78)}$$

and:

$$\frac{\partial Q_f}{\partial x_f} + \frac{\partial A_f}{\partial t} + \frac{\partial S}{\partial t} = q_c + q_l \qquad \text{(2-79)}$$

where the subscripts c and f refer to the channel and floodplain, respectively, q_l is the lateral inflow per unit length of floodplain, and q_c and q_f are the exchanges of water between the channel and the floodplain.

Equations 2-78 and 2-79 are now approximated using implicit finite differences by applying Equations 2-74 through 2-76:

$$\frac{\Delta Q_c}{\Delta x_c} + \frac{\Delta A_c}{\Delta t} = \bar{q}_f \tag{2-80}$$

$$\frac{\Delta Q_f}{\Delta x_c} + \frac{\Delta A_c}{\Delta t} + \frac{\Delta S}{\Delta t} = \bar{q}_c + \bar{q}_l \tag{2-81}$$

The exchange of mass is equal but not opposite in sign such that $\Delta x_c q_c = -q_f \Delta x_f$. Adding the above equations together and rearranging yields:

$$\Delta Q + \frac{\Delta A_c}{\Delta t}\Delta x_c + \frac{\Delta A_f}{\Delta t}\Delta x_f + \frac{\Delta S}{\Delta t}\Delta x_f - \bar{Q}_l = 0 \tag{2-82}$$

where $\bar{Q}_l$ is the average lateral inflow.

Momentum Equation

The momentum equation states that the rate of change in momentum is equal to the external forces acting on the system. From Appendix A, for a single channel:

$$\frac{\partial Q}{\partial t} + \frac{\partial (VQ)}{\partial x} + gA\left(\frac{\partial z}{\partial x} + S_f\right) = 0 \tag{2-83}$$

where: g = acceleration of gravity,
S_f = friction slope,
V = velocity.

The above equation can be written for the channel and for the floodplain:

$$\frac{\partial Q_c}{\partial t} + \frac{\partial (V_c Q_c)}{\partial x_c} + gA_c\left(\frac{\partial z}{\partial x_c} + S_{fc}\right) = M_f \tag{2-84}$$

$$\frac{\partial Q_f}{\partial t} + \frac{\partial (V_f Q_f)}{\partial x_f} + gA_f\left(\frac{\partial z}{\partial x_f} + S_{ff}\right) = M_c \tag{2-85}$$

where M_c and M_f are the momentum fluxes per unit distance exchanged between the channel and floodplain, respectively. Note that in Equations 2-84 and 2-85 the water surface elevation is not subscripted. An assumption in these equations is that the water surface is horizontal at any cross section perpendicular to the flow. Therefore, the water surface elevation is the same for the channel and the floodplain at a given cross section.

Using Equations 2-74 through 2-76, the above equations are approximated using finite differences:

$$\frac{\Delta Q_c}{\Delta t} + \frac{\Delta(V_c Q_c)}{\Delta x_c} + g\overline{A}_c\left(\frac{\Delta z}{\Delta x_c} + \overline{S}_{fc}\right) = M_f \qquad (2\text{-}86)$$

$$\frac{\Delta Q_f}{\Delta t} + \frac{\Delta(V_f Q_f)}{\Delta x_f} + g\overline{A}_f\left(\frac{\Delta z}{\Delta x_f} + \overline{S}_{ff}\right) = M_c \qquad (2\text{-}87)$$

Note that $\Delta x_c M_c = -\Delta x_f M_f$.

Adding and rearranging the above equations yields:

$$\frac{\Delta(Q_c \Delta x_c + Q_f \Delta x_f)}{\Delta t} + \Delta(V_c Q_c) + \Delta(V_f Q_f) + g(A_c + A_f)\Delta z + g\overline{A}_c \overline{S}_{fc}\Delta x_c + g\overline{A}_f \overline{S}_{ff}\Delta x_f = 0 \qquad (2\text{-}88)$$

The final two terms define the friction force from the banks acting on the fluid. An equivalent force can be defined as:

$$g\overline{A}\,\overline{S}_f \Delta x_e = g\overline{A}_c \overline{S}_{fc}\Delta x_c + g\overline{A}_f \overline{S}_{ff}\Delta x_f \qquad (2\text{-}89)$$

where: Δx_e = equivalent flow path,
S_f = friction slope for the entire cross section,
A = $\overline{A}_c + \overline{A}_f$.

Now, the convective terms can be rewritten by defining a velocity distribution factor:

$$\beta = \frac{(V_c^2 A_c + V_f^2 A_f)}{V^2 A} = \frac{(V_c Q_c + V_f Q_f)}{QV} \qquad (2\text{-}90)$$

then:

$$\Delta(\beta VQ) = \Delta(V_c Q_c) + \Delta(V_f Q_f) \qquad (2\text{-}91)$$

The final form of the momentum equation is:

$$\frac{\Delta(Q_c \Delta x_c + Q_f \Delta x_f)}{\Delta t} + \Delta(\beta VQ) + g\bar{A}\Delta z + g\bar{A}\bar{S}_f \Delta x_e = 0 \qquad (2\text{-}92)$$

A more familiar form is obtained by dividing through by Δx_e:

$$\frac{\Delta(Q_c \Delta x_c + Q_f \Delta x_f)}{\Delta t \Delta x_e} + \frac{\Delta(\beta VQ)}{\Delta x_e} + g\bar{A}(\frac{\Delta z}{\Delta x_e} + \bar{S}_f) = 0 \qquad (2\text{-}93)$$

Added Force Term

The friction and pressure forces from the banks do not always describe all the forces that act on the water. Structures such as bridge piers, navigation dams, and cofferdams constrict the flow and exert additional forces, which oppose the flow. In localized areas these forces can predominate and produce a significant increase in water surface elevation (called a "swell head") upstream of the structure.

For a differential distance, *dx*, the additional forces in the contraction produce a swell head of dh_l. This swell head is only related to the additional forces. The rate of energy loss can be expressed as a local slope:

$$S_h = \frac{dh_l}{dx} \qquad (2\text{-}94)$$

The friction slope in Equation 2-93 can be augmented by this term:

$$\frac{\partial Q}{\partial t} + \frac{\partial (VQ)}{\partial x} + gA(\frac{\partial z}{\partial x} + S_f + S_h) = 0 \qquad (2\text{-}95)$$

For steady flow, there are a number of relationships for computation of the swell head upstream of a contraction. For navigation dams, the formulas of Kindsvater and Carter, d'Aubuisson (Chow, 1959), and Nagler were reviewed by Denzel (1961). For bridges, the formulas of Yarnell (WES, 1973) and the Federal Highway Administration (FHWA, 1978) can be used. These formulas were all determined by experimentation and can be expressed in the more general form:

$$h_l = C\frac{V^2}{2g} \qquad (2\text{-}96)$$

where h_l is the head loss and C is a coefficient. The coefficient C is a function of velocity, depth, and the geometric properties of the opening, but for simplicity, it is assumed to be a constant. The location where the velocity head is evaluated varies from method to method. Generally, the velocity head is evaluated at the tailwater for tranquil flow and at the headwater for supercritical flow in the contraction.

If h_l occurs over a distance Δx_e, then $h_l = \overline{S}_h \Delta x_e$ and $\overline{S}_h = h_l / \Delta x_e$ where $\overline{S}_h$ is the average slope over the interval Δx_e. Within HEC-RAS, the steady flow bridge and culvert routines are used to compute a family of rating curves for the structure. During the simulation, for a given flow and tailwater, a resulting headwater elevation is interpolated from the curves. The difference between the headwater and tailwater is set to h_l and then $\overline{S}_h$ is computed. The result is inserted in the finite difference form of the momentum equation (Equation 2-93), yielding:

$$\frac{\Delta(Q_c \Delta x_c + Q_f \Delta x_f)}{\Delta t \Delta x_e} + \frac{\Delta(\beta VQ)}{\Delta x_e} + g\overline{A}\left(\frac{\Delta z}{\Delta x_e} + \overline{S}_f + \overline{S}_h\right) = 0 \qquad (2\text{-}97)$$

Lateral Influx of Momentum

At stream junctions, the momentum as well as the mass of the flow from a tributary enters the receiving stream. If this added momentum is not included in the momentum equation, the entering flow has no momentum and must be accelerated by the flow in the river. The lack of entering momentum causes the convective acceleration term, $\partial (VQ) / \partial x$, to become large. To balance the spatial change in momentum, the water surface slope must be large enough to provide the force to accelerate the fluid. Thus, the water surface has a drop across the reach where the flow enters creating backwater upstream of the junction on the main stem. When the tributary flow is large in relation to that of the receiving stream, the momentum exchange may be significant. The confluence of the Mississippi and Missouri Rivers is such a juncture. During a large flood, the computed decrease in water surface elevation over the Mississippi reach is over 0.5 feet if the influx of momentum is not properly considered.

The entering momentum is given by:

$$M_l = \xi \frac{Q_l V_l}{\Delta x} \qquad (2\text{-}98)$$

where: Q_l = lateral inflow,
V_l = average velocity of lateral inflow,
ξ = fraction of the momentum entering the receiving stream.

The entering momentum is added to the right side of Equation 2-97, hence:

$$\frac{\Delta(Q_c \Delta x_c + Q_f \Delta x_f)}{\Delta t \Delta x_e} + \frac{\Delta(\beta VQ)}{\Delta x_e} + g\bar{A}\left(\frac{\Delta z}{\Delta x_e} + \bar{S}_f + \bar{S}_h\right) = \xi \frac{Q_l V_l}{\Delta x_e} \qquad (2\text{-}99)$$

Equation 2-99 is only used at stream junctions in a dendritic model.

Finite Difference Form of the Unsteady Flow Equations

Equations 2-77 and 2-83 are nonlinear. If the implicit finite difference scheme is directly applied, a system of nonlinear algebraic equations results. Amain and Fang (1970), Fread (1974, 1976) and others have solved the nonlinear equations using the Newton-Raphson iteration technique. Apart from being relatively slow, that iterative scheme can experience troublesome convergence problems at discontinuities in the river geometry. To avoid the nonlinear solution, Preissmann (as reported by Liggett and Cunge, 1975) and Chen (1973) developed a technique for linearizing the equations. The following section describes how the finite difference equations are linearized in HEC-RAS.

Linearized, Implicit, Finite Difference Equations

The following assumptions are applied:

1. If $f \bullet f >> \Delta f \bullet \Delta f$, then $\Delta f \bullet \Delta f = 0$ (Preissmann as reported by Liggett and Cunge, 1975).

2. If g = g(Q,z), then Δg can be approximated by the first term of the Taylor Series, i.e.:

$$\Delta g_j = \left(\frac{\partial g}{\partial Q}\right)_j \Delta Q_j + \left(\frac{\partial g}{\partial z}\right)_j \Delta z_j \qquad (2\text{-}100)$$

3. If the time step, Δt, is small, then certain variables can be treated explicitly; hence $h_j^{n+1} \approx h_j^n$ and $\Delta h_j \approx 0$.

Assumption 2 is applied to the friction slope, S_f and the area, A. Assumption 3 is applied to the velocity, V, in the convective term; the velocity distribution factor, β; the equivalent flow path, x; and the flow distribution factor, ϕ.

The finite difference approximations are listed term by term for the continuity equation in Table 2-1 and for the momentum equation in Table 2-2. If the unknown values are grouped on the left-hand side, the following linear equations result:

$$CQ1_j \Delta Q_j + CZ1_j \Delta z_j + CQ2_j \Delta Q_{j+1} + CZ2_j \Delta z_{j+1} = CB_j \tag{2-101}$$

$$MQ1_j \Delta Q_j + MZ1_j \Delta z_j + MQ2_j \Delta Q_{j+1} + MZ2_j \Delta z_{j+1} = MB_j \tag{1-102}$$

Table 2-1
Finite Difference Approximation of the Terms in the Continuity Equation

Term	Finite Difference Approximation
ΔQ	$(Q_{j+1} - Q_j) + \theta(\Delta Q_{j+1} - \Delta Q_j)$
$\frac{\partial A_c}{\partial t}\Delta x_c$	$0.5\Delta x_{cj}\frac{\left(\frac{dA_c}{dz}\right)_j \Delta z_j + \left(\frac{dA_c}{dz}\right)_{j+1} \Delta z_{j+1}}{\Delta t}$
$\frac{\partial A_f}{\partial t}\Delta x_f$	$0.5\Delta x_{fj}\frac{\left(\frac{dA_f}{dz}\right)_j \Delta z_j + \left(\frac{dA_f}{dz}\right)_{j+1} \Delta z_{j+1}}{\Delta t}$
$\frac{\partial S}{\partial t}\Delta x_f$	$0.5\Delta x_{fj}\frac{\left(\frac{dS}{dz}\right)_j \Delta z_j + \left(\frac{dS}{dz}\right)_{j+1} \Delta z_{j+1}}{\Delta t}$

Table 2-2
Finite Difference Approximation of the Terms in the Momentum Equation

Term	***Finite Difference Approximation***
$\frac{\partial(Q_c \Delta x_c + Q_f \Delta x_f)}{\partial t \Delta x_e}$	$\frac{0.5}{\Delta x_e \partial t}(\partial Q_{cj} \Delta x_{cj} + \partial Q_{fj} \Delta x_{fj} + \partial Q_{cj+1} \Delta x_{cj} + \partial Q_{fj+1} \Delta x_{fj})$
$\frac{\Delta \beta VQ}{\Delta x_{ej}}$	$\frac{1}{\Delta x_{ej}}\left[(\beta VQ)_{j+1} - (\beta VQ)_j\right] + \frac{\theta}{\Delta x_{ej}}\left[(\beta VQ)_{j+1} - (\beta VQ)_j\right]$
$g\overline{A}\frac{\Delta z}{\Delta x_e}$	$g\overline{A}\left[\frac{z_{j+1} - z_j}{\Delta x_{ej}} + \frac{\theta}{\Delta x_{ej}}(\Delta z_{j+1} - \Delta z_j)\right] + \theta g \Delta \overline{A}\frac{(z_{j+1} - z_j)}{\Delta x_{ej}}$
$g\overline{A}(\overline{S}_f + \overline{S}_h)$	$g\overline{A}(\overline{S}_f + \overline{S}_h) + 0.5\theta g\overline{A}\left[(\Delta S_{fj+1} + \Delta S_{fj}) + (\Delta S_{hj+1} + \Delta S_{hj})\right] + 0.5\theta g(\overline{S}_f + \overline{S}_h)(\Delta A_j + \Delta A_{j+1})$
$\overline{A}$	$0.5(A_{j+1} + A_j)$
$\overline{S}_f$	$0.5(S_{fj+1} + S_{fj})$
∂A_j	$\left(\frac{dA}{dZ}\right)_j \Delta z_j$
∂S_{fj}	$\left(\frac{-2 S_f}{K}\frac{dK}{dz}\right)_j \Delta z_j + \left(\frac{2 S_f}{Q}\right)_j \Delta Q_j$
$\partial \overline{A}$	$0.5(\Delta A_j + \Delta A_{j+1})$

The values of the coefficients are defined in Tables 2-3 and 2-4.

Table 2-3
Coefficients for the Continuity Equation

Coefficient	**Value**
$CQ1_j$	$\dfrac{-\theta}{\Delta x_{ej}}$
$CZ1_j$	$\dfrac{0.5}{\Delta t \Delta x_{ej}}\left[\left(\dfrac{dA_c}{dz}\right)_j \Delta x_{cj} + \left(\dfrac{dA_f}{dz} + \dfrac{dS}{dz}\right)_j \Delta x_{fj}\right]$
$CQ2_j$	$\dfrac{\theta}{\Delta x_{ej}}$
$CZ2_j$	$\dfrac{0.5}{\Delta t \Delta x_{ej}}\left[\left(\dfrac{dA_c}{dz}\right)_{j+1} \Delta x_{cj} + \left(\dfrac{dA_f}{dz} + \dfrac{dS}{dz}\right)_{j+1} \Delta x_{fj}\right]$
CB_j	$-\dfrac{Q_{j+1} - Q_j}{\Delta x_{ej}} + \dfrac{Q_l}{\Delta x_{ej}}$

Table 2-4

Coefficients of the Momentum Equation

Term	Value
$MQ1_j$	$0.5\frac{\Delta x_{cj}\phi_j + \Delta x_{fj}(1-\phi_j)}{\Delta x_{ej}\Delta t} - \frac{\beta_j V_j \theta}{\Delta x_{ej}} + \theta g\overline{A}\frac{(S_{fj} + S_{hj})}{Q_j}$
$MZ1_j$	$\frac{-gA\theta}{\Delta x_{ej}} + 0.5g(z_{j+1} - z_j)\left(\frac{dA}{dz}\right)_j\left(\frac{\theta}{\Delta x_{ej}}\right) - g\theta\overline{A}\left[\left(\frac{dK}{dz}\right)_j\left(\frac{S_{fj}}{K_j}\right) + \left(\frac{dA}{dz}\right)_j\left(\frac{S_{hj}}{A_j}\right)\right] + 0.5\theta g\left(\frac{dA}{dz}\right)_j(\overline{S}_f + \overline{S}_h)$
$MQ2_j$	$0.5\left[\Delta x_{cj}\phi_{j+1} + \Delta x_{fj}(1-\phi_{j+1})\right]\left(\frac{1}{\Delta x_{ej}\Delta t}\right) + \beta_{j+1}V_{j+1}\left(\frac{\theta}{\Delta x_{ej}}\right) + \frac{\theta gA}{Q_{j+1}}(S_{fj+1} + S_{hj+1})$
$MZ2_j$	$\frac{g\overline{A}\theta}{\Delta x_{ej}} + 0.5g(z_{j+1} - z_j)\left(\frac{dA}{dz}\right)_{j+1}\left(\frac{\theta}{\Delta x_{ej}}\right) - \theta g\overline{A}\left[\left(\frac{dK}{dz}\right)_{j+1}\left(\frac{S_{fj+1}}{K_{j+1}}\right) + \left(\frac{dA}{dz}\right)_{j+1} + \left(\frac{S_{hj+1}}{A_{j+1}}\right)\right] + 0.5\theta g\left(\frac{dA}{dz}\right)_{j+1}(\overline{S}_f + \overline{S}_h)$
MB_j	$-\left[(\beta_{j+1}V_{j+1}Q_{j+1} - \beta_j V_j Q_j)\left(\frac{1}{\Delta x_{ej}}\right) + \left(\frac{g\overline{A}}{\Delta x_{ej}}\right)(z_{j+1} - z_j) + g\overline{A}(\overline{S}_f + \overline{S}_h)\right]$

Flow Distribution Factor

The distribution of flow between the channel and floodplain must be determined. The portion of the flow in the channel is given by:

$$\phi_j = \frac{Q_{cj}}{Q_{cj} + Q_{fj}} \tag{2-103}$$

Fread (1976) assumed that the friction slope is the same for the channel and floodplain, thus the distribution is given by the ratio of conveyance:

$$\phi_j = \frac{K_{cj}}{K_{cj} + K_{fj}} \tag{2-104}$$

Equation 2-104 is used in the HEC-RAS model.

Equivalent Flow Path

The equivalent flow path is given by:

$$\Delta x_e = \frac{\overline{A}_c \overline{S}_{fc} \Delta x_c + \overline{A}_f \overline{S}_{ff} \Delta x_f}{\overline{A}\,\overline{S}_f} \tag{2-105}$$

If we assume:

$$\overline{\phi} = \frac{\overline{K}_c}{\overline{K}_c + \overline{K}_f} \tag{2-106}$$

where $\overline{\phi}$ is the average flow distribution for the reach, then:

$$\Delta x_e = \frac{\overline{A}_c \Delta x_c + \overline{A}_f \Delta x_f}{\overline{A}} \tag{2-107}$$

Since Δx_e is defined explicitly:

$$\Delta x_{ej} = \frac{(A_{cj} + A_{cj+1}) \Delta x_{cj} + (A_{fj} + A_{fj+1}) \Delta x_{fj}}{A_j + A_{j+1}} \tag{2-108}$$

Boundary Conditions

For a reach of river there are N computational nodes which bound N-1 finite difference cells. From these cells 2N-2 finite difference equations can be developed. Because there are 2N unknowns (ΔQ and Δz for each node), two additional equations are needed. These equations are provided by the boundary conditions for each reach, which for subcritical flow, are required at the upstream and downstream ends. For supercritical flow, boundary conditions are only required at the upstream end.

Interior Boundary Conditions (for Reach Connections)

A network is composed of a set of M individual reaches. Interior boundary equations are required to specify connections between reaches. Depending on the type of reach junction, one of two equations is used:

Continuity of flow:

$$\sum_{i=1}^{l} S_{gi} Q_i = 0 \qquad (2\text{-}109)$$

where: l = the number of reaches connected at a junction,
S_{gi} = -1 if i is a connection to an upstream reach, +1 if i is a connection to a downstream reach,
Q_i = discharge in reach i.

The finite difference form of Equation 2-109 is:

$$\sum_{i=1}^{l-1} MU_{mi} \Delta Q_i + MUQ_m \Delta Q_K = MUB_m \qquad (2\text{-}110)$$

where: MU_{mi} = $\theta\, S_{gi}$,
MUQ_m = $\theta\, S_{gK}$,
MUB_m = $- \sum_{i=1}^{l} S_{gi} Q_i$

Continuity of stage:

$$z_k = z_c \quad (2\text{-}111)$$

where z_k, the stage at the boundary of reach k, is set equal to z_c, a stage common to all stage boundary conditions at the junction of interest. The finite difference form of Equation 2-111 is:

$$MUZ_m \Delta z_K - MU_m \Delta z_c = MUB_m \quad (2\text{-}112)$$

where: $MUZ_m = 0$,
$MU_m = 0$,
$MUB_m = z_c - z_K$.

With reference to Figure 2.12, HEC-RAS uses the following strategy to apply the reach connection boundary condition equations:

- Apply flow continuity to reaches upstream of flow splits and downstream of flow combinations (reach 1 in Figure 2.12). Only one flow boundary equation is used per junction.

- Apply stage continuity for all other reaches (reaches 2 and 3 in Figure 2.12). Z_c is computed as the stage corresponding to the flow in reach 1. Therefore, stage in reaches 2 and 3 will be set equal to Z_c.

Upstream Boundary Conditions

Upstream boundary conditions are required at the upstream end of all reaches that are not connected to other reaches or storage areas. An upstream boundary condition is applied as a flow hydrograph of discharge versus time. The equation of a flow hydrograph for reach m is:

$$\Delta Q_k^{n+1} = Q_k^n - Q_k \quad (2\text{-}113)$$

where k is the upstream node of reach m. The finite difference form of Equation 2-113 is:

$$MUQ_m \Delta dQ_K = MUB_m \quad (2\text{-}114)$$

where: $MUQ_m = 1$,
$MUB_m = Q_l^{n+1} - Q_l^n$.

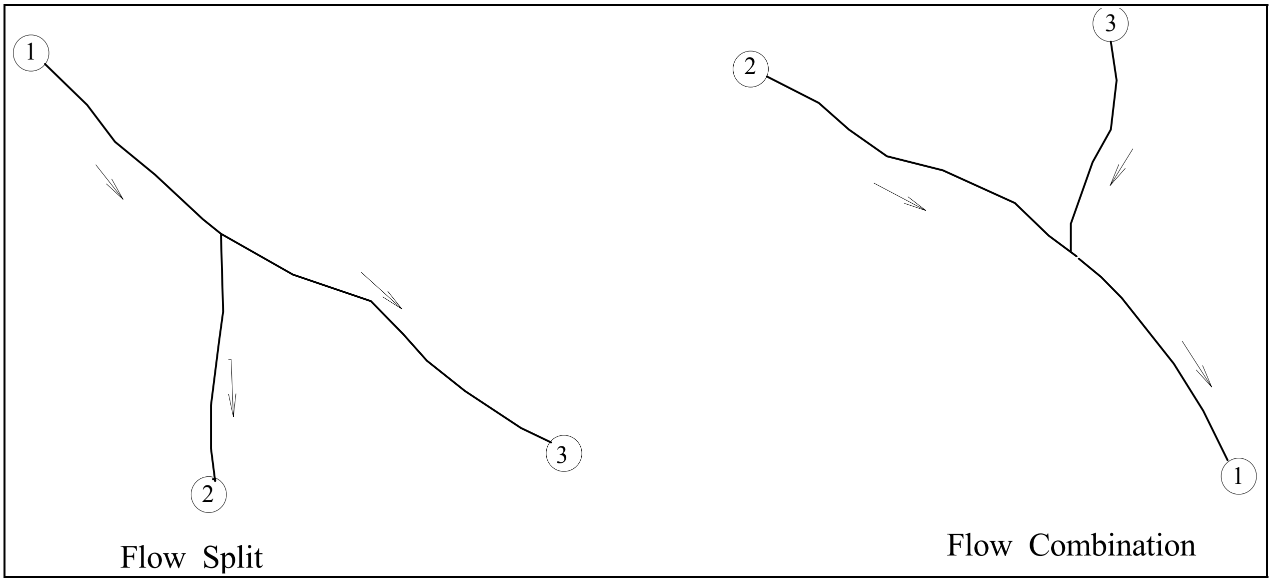


Figure 2.12 Typical flow split and combination.

Downstream Boundary Conditions

Downstream boundary conditions are required at the downstream end of all reaches which are not connected to other reaches or storage areas. Four types of downstream boundary conditions can be specified:

- a stage hydrograph,
- a flow hydrograph,
- a single-valued rating curve,
- normal depth from Manning's equation.

Stage Hydrograph. A stage hydrograph of water surface elevation versus time may be used as the downstream boundary condition if the stream flows into a backwater environment such as an estuary or bay where the water surface elevation is governed by tidal fluctuations, or where it flows into a lake or reservoir of known stage(s). At time step (n+1)Δt, the boundary condition from the stage hydrograph is given by:

$$\Delta Z_N = Z_N^{n+1} - Z_N^n \qquad (2\text{-}115)$$

The finite difference form of Equation 2-115 is:

$$CDZ_m \Delta z_N = CDB_m \qquad (2\text{-}116)$$

where: CDZ_m = 1,
CDB_m = $z_N^{n+1} - z_N^n$.

<u>Flow Hydrograph.</u> A flow hydrograph may be used as the downstream boundary condition if recorded gage data is available and the model is being calibrated to a specific flood event. At time step (n+1)Δt, the boundary condition from the flow hydrograph is given by the finite difference equation:

$$CDQ_m \Delta Q_N = CDB_m \qquad (2\text{-}117)$$

where: CDQ_m = 1,
CDB_m = $Q_N^{n+1} - Q_N^n$.

<u>Single Valued Rating Curve.</u> The single valued rating curve is a monotonic function of stage and flow. An example of this type of curve is the steady, uniform flow rating curve. The single valued rating curve can be used to accurately describe the stage-flow relationship of free outfalls such as waterfalls, or hydraulic control structures such as spillways, weirs or lock and dam operations. When applying this type of boundary condition to a natural stream, caution should be used. If the stream location would normally have a looped rating curve, then placing a single valued rating curve as the boundary condition can introduce errors in the solution. Too reduce errors in stage, move the boundary condition downstream from your study area, such that it no longer affects the stages in the study area. Further advice is given in (USACE, 1993).

At time (n+1)Δt the boundary condition is given by:

$$Q_N + \theta \Delta Q_N = D_{k-1} + \frac{D_k - D_{k-1}}{S_k - S_{k-1}} (z_N + \Delta z_N - S_{k-1}) \qquad (2\text{-}118)$$

where: D_k = K^{th} discharge ordinate,
S_k = K^{th} stage ordinate.

After collecting unknown terms on the left side of the equation, the finite difference form of Equation 2-118 is:

$$CDQ_m \Delta Q_N + CDZ_m \Delta z_N = CDB_m \qquad (2\text{-}119)$$

where: $CDQ_m = \theta,$

$$CDZ_m = \frac{D_k - D_{k-1}}{S_k - S_{k-1}},$$

$$CDB_m = Q_N + D_{k-1} + \frac{D_k - D_{k-1}}{S_k - S_{k-1}}(z_N - S_{k-1}).$$

Normal Depth. Use of Manning's equation with a user entered friction slope produces a stage considered to be normal depth if uniform flow conditions existed. Because uniform flow conditions do not normally exist in natural streams, this boundary condition should be used far enough downstream from your study area that it does not affect the results in the study area. Manning's equation may be written as:

$$Q = K(S_f)^{0.5} \qquad (2\text{-}120)$$

where: K represents the conveyance and S_f is the friction slope.

Skyline Solution of a Sparse System of Linear Equations

The finite difference equations along with external and internal boundary conditions and storage area equations result in a system of linear equations which must be solved for each time step:

$$Ax = b \qquad (2\text{-}121)$$

in which: A = coefficient matrix,
x = column vector of unknowns,
b = column vector of constants.

For a single channel without a storage area, the coefficient matrix has a band width of five and can be solved by one of many banded matrix solvers.

For network problems, sparse terms destroy the banded structure. The sparse terms enter and leave at the boundary equations and at the storage areas. Figure 2.13 shows a simple system with four reaches and a storage area off of reach 2. The corresponding coefficient matrix is shown in Figure 2.14. The elements are banded for the reaches but sparse elements appear at the reach boundaries and at the storage area. This small system is a trivial problem to solve, but systems with hundreds of cross sections and tens of reaches pose a major numerical problem because of the sparse terms. Even the largest computers cannot store the coefficient matrix for a moderately sized problem, furthermore, the computer time required to solve such a large matrix using Gaussian elimination would be very large. Because most of the elements are zero, a majority of computer time would be wasted.

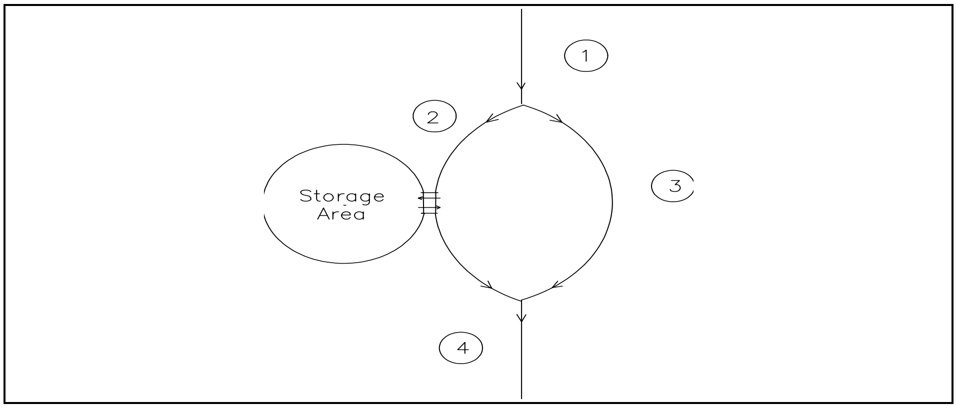


Figure 2.13 Simple network with four reaches and a storage area.

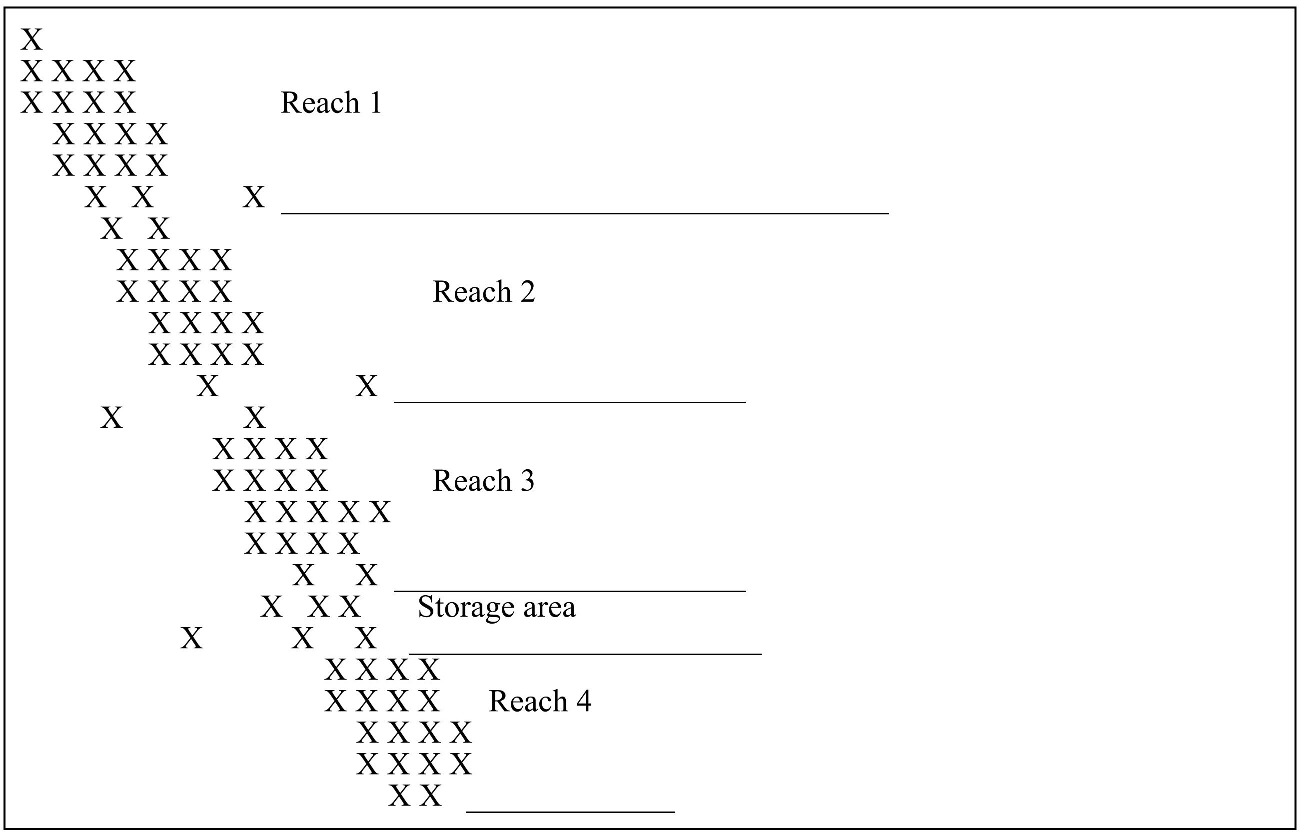


Figure 2.14 Sparse coefficient matrix resulting from simple linear system. Note, sparse terms enter and disappear at storage areas and boundary equations.

Three practical solution schemes have been used to solve the sparse system of linear equations: Barkau (1985) used a front solver scheme to eliminate terms to the left of the diagonal and pointers to identify sparse columns to the right of the diagonal. Cunge et al. (1980) and Shaffranekk (1981) used recursive schemes to significantly reduce the size of the sparse coefficient matrix. Tucci (1978) and Chen and Simons (1979) used the skyline storage scheme (Bathe and Wilson, 1976) to store the coefficient matrix. The goal of these schemes is to more effectively store the coefficient matrix. The front solver and skyline methods identify and store only the significant elements. The recursive schemes are more elegant, significantly reducing the number of linear equations. All use Gaussian elimination to solve the simultaneous equations.

A front solver performs the reduction pass of Gauss elimination before equations are entered into a coefficient matrix. Hence, the coefficient matrix is upper triangular. To further reduce storage, Barkau (1985) proposed indexing sparse columns to the right of the band, thus, only the band and the sparse terms were stored. Since row and column operations were minimized, the procedure should be as fast if not faster than any of the other procedures. But, the procedure could not be readily adapted to a wide variety of problems because of the way that the sparse terms were indexed. Hence, the program needed to be re-dimensioned and recompiled for each new problem.

The recursive schemes are ingenious. Cunge credits the initial application to Friazinov (1970). Cunge's scheme and Schaffranek's scheme are similar in approach but differ greatly in efficiency. Through recursive upward and downward passes, each single routing reach is transformed into two transfer equations which relate the stages and flows at the upstream and downstream boundaries. Cunge substitutes the transfer equations in which M is the number of junctions. Schraffranek combines the transfer equations with the boundary equations, resulting in a system of $4N$ equations in which N is the number of individual reaches. The coefficient matrix is sparse, but the degree is much less than the original system.

By using recursion, the algorithms minimize row and column operations. The key to the algorithm's speed is the solution of a reduced linear equation set. For smaller problems Gaussian elimination on the full matrix would suffice. For larger problems, some type of sparse matrix solver must be used, primarily to reduce the number of elementary operations. Consider, for example, a system of 50 reaches. Schaffranek's matrix would be 200 X 200 and Cunge's matrix would be 50 X 50, 2.7 million and 42,000 operations respectively (the number of operations is approximately 1/3 n^3 where n is the number of rows).

Another disadvantage of the recursive scheme is adaptability. Lateral weirs which discharge into storage areas or which discharge into other reaches disrupt the recursion algorithm. These weirs may span a short distance or they may span an entire reach. The recursion algorithm, as presented in the above references, will not work for this problem. The algorithm can be adapted, but no documentation has yet been published.

Skyline is the name of a storage algorithm for a sparse matrix. In any sparse matrix, the non-zero elements from the linear system and from the Gaussian elimination procedure are to the left of the diagonal and in a column above the diagonal. This structure is shown in Figure A.4. Skyline stores these inverted "L shaped" structures in a vector, keeping the total storage at a minimum. Elements in skyline storage are accessed by row and column numbers. Elements outside the "L" are returned as zero, hence the skyline matrix functions exactly as the original matrix. Skyline storage can be adapted to any problem.

The efficiency of Gaussian elimination depends on the number of pointers into skyline storage. Tucci (1978) and Chen and Simons (1979) used the original algorithm as proposed by Bathe and Wilson (1976). This algorithm used only two pointers, the left limit and the upper limit of the "L", thus, a large number of unnecessary elementary operations are performed on zero elements and in searching for rows to reduce. Their solution was acceptable for small problems, but clearly deficient for large problems. Using additional pointers reduces the number of superfluous calculations. If the pointers identify all the sparse columns to the right of the diagonal, then the number of operations is minimized and the performance is similar to the front solver algorithm.

<u>Skyline Solution Algorithm</u>

The skyline storage algorithm was chosen to store the coefficient matrix. The Gauss elimination algorithm of Bathe and Wilson was abandoned because of its poor efficiency. Instead a modified algorithm with seven pointers was developed. The pointers are:

1) IDIA(IROW) - index of the diagonal element in row IROW in skyline storage.
2) ILEFT(IROW) - number of columns to the left of the diagonal.
3) IHIGH(IROW) - number of rows above the diagonal.
4) IRIGHT(IROW) - number of columns in the principal band to the right of the diagonal.
5) ISPCOL(J,IROW) - pointer to sparse columns to the right of the principal band.
6) IZSA(IS) - the row number of storage area IS.
7) IROWZ(N) - the row number of the continuity equation for segment N.

The pointers eliminate the meaningless operations on zero elements. This code is specifically designed for flood routing through a full network.